we substitute into it the expressions

$$\cos\varphi = \frac{t}{\sqrt{t^2+1}}, \quad \sin\varphi = \frac{1}{\sqrt{t^2+1}}, \quad \cos 2k\varphi = \operatorname{Re}\left(\frac{t+i}{t-i}\right)^k$$

Formula (5.8) may be rewritten in the form

$$H = -\sum_{k=0}^{\infty} J_{2k}(\beta x) S_k(y,z), \quad S_0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\chi(z+yt)+\chi(-z-yt)}{t^2+1}\,dt$$

$$S_k(y,z) = \frac{1}{\pi}\operatorname{Re}\int_{-\infty}^{+\infty} \chi(z+yt)\frac{(t+i)^{k-1}}{(t-i)^{k+1}}\,dt, \quad k \geqslant 1$$

Using the theory of residues to evaluate the integrals, we obtain (2.3).

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LONG-WAVE THERMOCAPILLARY CONVECTION IN LAYERS WITH DEFORMABLE INTERFACES*

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Using non-linear equations describing finite-amplitude deformation of the interfaces /1/ of a system of horizontal immiscible liquid layers, long-wave convective flows are studied for nearly critical Marangoni numbers. The distortion of the interfaces is assumed to be weak. Approximate evolution equations are obtained for the deformation of the interfaces. Analytic solutions describing the stationary surface profile for thermocapillary convection are found, and their stability is investigated.

1. Suppose two horizontal solid plates $(z=0, z=a)$ are maintained at constant and different temperatures, (the temperature difference being equal to θ), and that the space between the plates is filled with two immiscible liquid layers. In equilibrium the thickness of the lower (second) layer is equal to Ha, and that of the higher (first) layer is $(1-H)a$; $0<H<1$. The densities of the media, the coefficients of dynamic and kinematic viscosity, and the thermal conductivity and thermal diffusivity are equal to $\rho_m, \eta_m, \nu_m, \varkappa_m$, and χ_m $(m=1$ for the upper layer and $m=2$ for the lower layer). The surface tension σ depends linearly on temperature T: $\sigma = \sigma_0(1-\alpha T)$.

As units of length, time, velocity, pressure and temperature we take a, a^2/ν_1, ν_1/a, $\rho_1\nu_1^2/a^2$ and θ respectively. In dimensionless variables the convection equations and boundary conditions are written in the form

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$$D_1\bar{v}_1 = -\nabla p_1 + \Delta\bar{v}_1,\ \operatorname{div}\bar{v}_1 = 0,\ D_1T_1 = P^{-1}\Delta T_1 \tag{1.1}$$
$$D_2\bar{v}_2 = -\sigma\nabla p_2 + \nu^{-1}\Delta\bar{v}_2,\ \operatorname{div}\bar{v}_2 = 0,\ D_2T_2 = (\chi P)^{-1}\,\Delta T_2$$
$$z = 1:\ \bar{v}_1 = 0,\ T_1 = 0;\ z = 0:\ \bar{v}_2 = 0,\ T_2 = s$$
$$z = H + h:\ p_1 - p_2 + R^{-1}(W - MT) + Gh = S_{ik}n_in_k$$
$$S_{ik}\,\tau_i^{(l)}n_k - M\tau_i^{(l)}\partial T_1/\partial x_i = 0$$
$$v_1 = v_2,\ \partial h/\partial t + v_{1x}\partial h/\partial x + v_{1y}\partial h/\partial y = v_{1z}$$
$$T_1 = T_2,\ (\varkappa\partial T_1/\partial x_i - \partial T_2/\partial x_i)\,n_i = 0$$
$$P = \frac{\nu_1}{\chi_1},\quad G = \frac{ga^3\rho_1(\rho_2-\rho_1)}{\eta_1^2},\quad M = \frac{\alpha\theta a}{\eta_1\nu_1},\quad W = \frac{\sigma a}{\eta_1\nu_1}$$
$$D_m = \partial/\partial t + v_m\nabla,\ m = 1,\ 2$$
$$S_{ik} = (\partial v_{1i}/\partial x_k + \partial v_{1k}/\partial x_i) - \eta^{-1}(\partial v_{2i}/\partial x_k + \partial v_{2k}/\partial x_i)$$

Here v_m, p_m and T_m are the velocity, pressure and temperature in the m-th liquid, (the pressure is measured relative to the hydrostatic pressure), h is the displacement of the interface, n is the vector normal to the surface, $\tau^{(l)}$ $(l = 1, 2)$ are orthogonal tangent vectors, R is the radius of curvature of the surface, P is the Prandtl number, G is the modified Galileo number, M is an analogue of Marangoni's number, and the parameter s is equal to 1 for heating from below and -1 for heating from above.

When writing down the boundary-value problem (1.1) it was assumed that the Galileo numbers $G_m = ga_m^3/\nu_m^2$ $(m = 1, 2)$ are of the order of unity, and that the quantities $\beta_m\theta$ (where β_m is the coefficient of thermal expansion in the m-th liquid) are small. On this basis the Archimedean buoyancy force terms proportional to the Grashof number $\mathrm{Gr}_m = \beta_m\theta G_m$ are omitted.

The boundary-value problem (1.1) always has the solution

$$T_1^e = s(1 - z)I^{-1},\ T_2^e = s[(1 - H) + \varkappa(H - z)]I^{-1}\ (I = (1 - H) + \varkappa H)$$
$$v_1^e = 0,\ v_2^e = 0,\ p_1^e = 0,\ p^e_2 = 0,\ h^e = 0$$

corresponding to mechanical equilibrium.

The equilibrium can become unstable as M increases for some method of heating. In the long-wave domain the neutral curve is given by the expression /2/

$$M(k) = M_c + Nk^2 \tag{1.2}$$

The threshold value $M(0) = M_c$ and the heating method (the sign of s) for which long-wave instability occurs is given by the formula

$$sM_c = {}^2/_3G\varkappa^{-1}I^2JKH(1 - H) \tag{1.3}$$
$$J = [(1 - H) + \eta H],\ K = [(1 - H)^2 - \eta H^2]^{-1}$$

It can be seen that in the range $0 < H < H_* = 1/(1 + \sqrt{\eta})$ the instability appears for heating from below, and in the range $H_* < H < 1$ for heating from above. Within these ranges, and depending on the values of the parameters $\varkappa$ and η, the derivative $M_c' = dM_c/dH$ is either sign-constant or changes sign at the points $H = H_1$ and $H = H_2$. The region of the $\varkappa, \eta$ plane in which the function $M_c(H)$ is non-monotonic (Fig.1, curve *1*) consists of two subregions. One of them is shown by the hatched area on Fig.2 $(\varkappa_m = 2/3,\ \eta_m \simeq 0.1593)$; the other subregion is obtained by the transformation $\varkappa \to 1/\varkappa,\ \eta \to 1/\eta$. If the parameters $\varkappa$ and η lie on the boundary of the hatched region, the dependence of M_c on H has a point of inflection (Fig.1, curve *2*), at which both the derivatives M_c' and M_c'' vanish. Outside the hatched region M_c has no extrema and $M_c' > 0$ for all H (Fig.1, curve *3*).

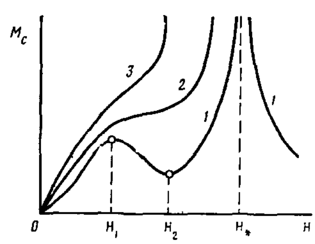

Fig.1

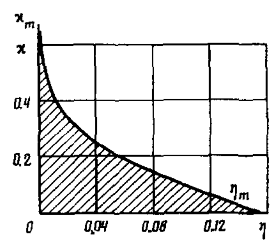

Fig.2

The quantity

$$N = M_c\{{}^1/_3(\varkappa - 1)\,I^{-1}H(1-H)(1-2H) + {}^1/_{15}[\eta(1-H) + H]\,J^{-1}H\cdot(1-H) + {}^2/_{15}(1-\eta)\,KH^2(1-H)^2 + WG^{-1} + {}^1/_{120}GP\eta KH^3(1-H)^3[(1-H)^2 - \chi H^2]\} \qquad (1.4)$$

can be either positive or negative, depending on the parameters of the system. In the latter case, however, long-wave perturbations are known not to be as dangerous, and so henceforth we shall consider the case $N > 0$.

2. Long-wave instability of the equilibrium leads to the development of thermocapillary flow, in which the amplitude of the distortion of the interface is, in general, of the order of the layer's thickness. This results in a rather complicated form for the governing equations /1/, the analysis of which is made more difficult.

We shall restrict ourselves to studying long-wave thermocapillary convection in the immediate neighbourhood of the threshold Marangoni number and for small displacements of the interface h. In this case the investigation of problem (1.1) is simplified.

Put

$$M = M_c + \varepsilon M^{(1)} \qquad (2.1)$$

where ε is a small parameter. We will assume the quantities W and G to be of the order of unity. According to (1.2) our principal interest $(N > 0)$ is in growing or weakly damped perturbations with wave numbers $k \sim \varepsilon^{1/2}$, and so it is advisable to perform a scale transformation

$$\bar{x} = \varepsilon^{1/2}x,\quad \bar{y} = \varepsilon^{1/2}y \qquad (2.2)$$

We will use the evolution time-scale for long-wave perturbations

$$\bar{t} = \varepsilon^2 t \qquad (2.3)$$

We restrict ourselves to considering a displacement h of the interface of order ε. We make the change of variables

$$h = \varepsilon\bar{h},\quad T_m = T_m^e + \varepsilon\theta_m,\quad p_m = \varepsilon P_m,$$
$$u_m = \varepsilon^{3/2}U_m,\quad v_m = \varepsilon^{3/2}V_m,\quad w_m = \varepsilon^2 W_m \quad (m = 1,\ 2) \qquad (2.4)$$

(where u, v, w denote the x-, y- and z-components of the velocity). The function h, θ_m, P_m, U_m, V_m and W_m are expanded in powers of ε:

$$h = h^{(0)} + \varepsilon h^{(1)} + \ldots \qquad (2.5)$$

and similarly for the other variables.

We substitute (2.1)-(2.5) into the full non-linear convection Eqs.(1.1) and equate terms of the same order in ε.

To lowest order in ε the solution has the form

$$\theta_1^{(0)} = h^{(0)}A_1(z-1),\quad P_1^{(0)} = h^{(0)}B_1 \qquad (2.6)$$
$$U_1^{(0)} = \frac{\partial h^{(0)}}{\partial \bar{x}}B_1\left[\frac{(z-1)^2}{2} + \frac{(1-H)(z-1)}{3}\right]$$
$$W_1^{(0)} = \Delta_2 h^{(0)}B_1\left[-\frac{(z-1)^3}{6} + \frac{(1-H)(z-1)^2}{6}\right]$$
$$\theta_2^{(0)} = h^{(0)}A_2 z,\quad P_2^{(0)} = h^{(0)}B_2$$
$$U_2^{(0)} = \frac{\partial h^{(0)}}{\partial \bar{x}}\eta B_2\left(\frac{z^2}{2} - \frac{Hz}{3}\right),\quad W_2^{(0)} = \Delta_2 h^{(0)}\eta B_2\left(-\frac{z^3}{6} + \frac{Hz}{6}\right),$$
$$A_1 = s(\varkappa - 1)\,I^{-2},\ B_1 = G\eta KH^2,\ A_2 = s\varkappa(\varkappa - 1)\,I^{-2},\ B_2 = GK(1-H)^2;$$
$$\bar{\Delta}_2 = \partial^2/\partial\bar{x}^2 + \partial^2/\partial\bar{y}^2$$

The expression for $V_m^{(0)}$ is obtained from the formulae for $U_m^{(0)}$ by changing $\partial/\partial\bar{x}$ to $\partial/\partial\bar{y}$. To lowest order the function $h^{(0)}(\bar{x}, \bar{y}, \bar{t})$ remains arbitrary.

From the conditions for the system of equations of next order in ε to be solvable we obtain an evolution equation for the function $h^{(0)}$, which reduces to the form

$$B\partial h^{(0)}/\partial\bar{t} = -M^{(1)}\bar{\Delta}_2 h^{(0)} = N\bar{\Delta}_2^2 h^{(0)} + {}^1/_2 M_c'\bar{\Delta}_2(h^{(0)})^2 \qquad (2.7)$$
$$B = 2s\varkappa^{-1}\eta^{-1}I^2K[K^{-2} + 4\eta H(1-H)]\,H^{-2}(1-H)^{-2}$$

Note that B is always positive, because, according to (1.3), the quantities s and K have the same sign. We recall that N is assumed to be positive, because in the opposite case long-wave perturbations are known not to be as dangerous (see formula (1.2)). As was remarked in Sect.1, everywhere except for discrete values of the parameter, $H = H_{1,2}$ (Fig.2). Here we assume that $M_c' \neq 0$; the case $M_c' = 0$ will be considered in Sect.3.

We introduce new variables

$$\tau = \frac{\bar{t}}{BN}, \quad X = \frac{\bar{x}}{N^{1/2}}; \quad Y = \frac{\bar{y}}{N^{1/2}}, \quad Z = -\frac{M_c'}{2} h^{(0)} \tag{2.8}$$

Then (2.6) reduces to the form

$$\partial Z/\partial\tau + \Delta_2^2 Z + M^{(1)}\Delta_2 Z + \Delta_2 (Z)^2 = 0 \tag{2.9}$$

A one-dimensional version of an equation like (2.9) was obtained in /3/ when investigating long-wave convective motions in anomalous thermocapillary situations.

We consider stationary one-dimensional flow regimes $(Z = Z(X))$ for which (2.9) takes the form

$$\frac{d^2}{dX^2}\left(\frac{d^2 Z}{dX^2} + M^{(1)} Z + Z^2\right) = 0 \tag{2.10}$$

The quantity Z is proportional to the displacement of the profile height from the mean position, and therefore satisfies the relation

$$\int_{-\infty}^{\infty} Z\, dX = 0 \tag{2.11}$$

Problem (2.10) has a family of periodic solutions

$$Z(X) = -\frac{M^{(1)}}{2} - \frac{2-q^2}{3q^2} A + \frac{1-q^2}{q^2} A\, \mathrm{dn}^{-2}\xi \tag{2.12}$$

$$A = \frac{3q^2 M^{(1)}}{2\left[q^2 - 2 + 3E(q)/K(q)\right]} > 0, \quad \xi = \frac{\sqrt{A}}{3q}(X - X_0) \tag{2.13}$$

where $\mathrm{dn}\,\xi$ is a Jacobi elliptic function with modulus q, $E(q)$ and $K(q)$ are complete elliptic integrals, and X_0 is an arbitrary constant. the spatial period of the function $Z(X)$ is

$$L = 2\pi/k, \quad k = \pi\sqrt{A/6}\,[qK(q)]^{-1} \tag{2.14}$$

We first consider the region $M^{(1)} > 0$ (i.e. $M > M_c$). The condition $A > 0$ is satisfied in this region when $0 < q < q_*$, where $q_* \simeq 0.98038$. The dependence of the profile amplitude A on the wave number k, specified by the parametric formulae (2.14) and (2.15), is shown in Fig.3 (the broken curve *1*). Branching occurs in the stable equilibrium region $k > k_* = [M^{(1)}]^{1/2}$, hence the stationary solutions of (2.11) are unstable. In the space of functions of prescribed period such a solution corresponds to a saddle point, whose stable manifold separates the regions of decaying and growing finite amplitude perturbations imposed on the equilibrium state.

The dependence of the parameter q on k is shown in Fig.3, (the solid curve *1*). We remark that even for small deviations k away from unity, the parameter q rapidly approaches q_*, so that the stationary profile is strongly non-sinusoidal.

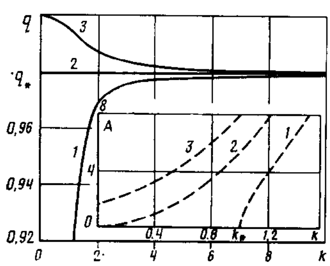

Fig.3

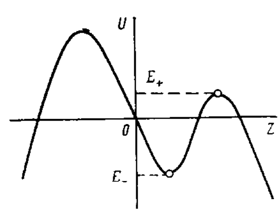

Fig.4

For $M^{(1)} = 0$ $(M = M_c)$, solution (2.12) satisfies condition (2.11) irrespective of the magnitude of A if $q = q_*$ (the solid curve *2* in Fig.3); in this case we have

$$A = 6\,(q_* K(q_*)/\pi)^2 k^2$$

(the broken curve *2* in Fig.3).

In the subcritical region $M^{(1)} < 0$ $(M < M_c)$, solution (2.12) exists for all values of the wave number k, with $q_* < q \leqslant 1$. The dependence of the amplitude A and the parameter q on the wave number k is shown in Fig.3 by the broken and solid curves 3 respectively. The minimum value of A is reached when $k \to 0$ $(q \to 1)$ for the solution

$$Z = {}^3/_2 \mid M^{(1)} \mid \mathrm{ch}^{-2} ({}^1/_2 \mid M^{(1)} \mid^{1/2} X)$$

The amplitude A, which determines the boundary of unstable equilibrium with respect to finite perturbations, increases rapidly as k increases, and because q stays close to unity, the form of the neutral finite-amplitude perturbation is strongly non-sinusoidal and resembles a chain of solitary waves. These properties would appear to explain the fact that finite-amplitude instability of equilibrium was not observed in numerical experiments /3/ when spatially periodic pertubations were imposed, while at the same time such instability was indeed observed for perturbations in the form of solitary waves.

Two-dimensional stationary solutions of (2.9) have not been constructed in analytic form. We will confine ourselves to analysing the branching of two-dimensional spatially periodic solutions in the form of rectangular and hexagonal cells for $M^{(1)} > 0$ in the neighbourhood of $k = k_*$. For rectangular cells a series expansion of (2.9) with respect to the amplitude gives the following approximate expression:

$$Z = [a(S^2)(k - k_*) k_*]^{1/2} \cos k\sqrt{1 - S^2} X \cos SkY + O(k - k_*)$$

$$a(S^2) = \frac{48(3 - 4S^2)(1 - 4S^2)}{16S^4 - 16S^2 - 9}, \quad S = \sin\frac{\varphi}{2}$$

Here φ is the angle between the basis wave vectors of the rectangular structure. For $\varphi < 60^\circ$ the branching is mild, but for $\varphi > 60^\circ$ it is severe. For a hexagonal structure the branching has a two-sided character:

$$Z = 2(k - k_*)\left(\cos kX + 2\cos\frac{kX}{2}\cos\frac{\sqrt{3}}{2}kY\right) + o(k - k_*)$$

As has already been noted, the stationary solution determines the threshold of equilibrium stability with respect to finite amplitude perturbations. It is clear that for small $k - k_*$ the amplitude of a finite perturbation in the form of hexagonal cells, leading to an instability of the equilibrium, is much smaller than the amplitude of the corresponding perturbation in the form of rollers.

3. We now consider the case when $M_c' = 0$. As was noted in Sect.1, this can arise for certain values $H = H_{1,2}$ if the parameters η and $\varkappa$ lie inside specified domains. We assume that instead of (2.1)

$$h = \varepsilon^{1/2}\bar{h}, \; T_m = T_m^e + \varepsilon^{1/2}\theta_m, \; p_m = \varepsilon^{1/2} P_m$$

$$u_m = \varepsilon U_m, \; v_m = \varepsilon V_m, \; w_m = \varepsilon^{3/2} W_m$$

and expand the variables as power series in $\varepsilon^{1/2}$. In the zeroth order we again obtain solution (2.6); the condition of solvability of the order $\varepsilon^{1/2}$ problem reduces to $M_c' = 0$. Finally, at order ε we obtain the evolution equation for $h^{(0)}$:

$$B\frac{\partial h^{(0)}}{\partial t} = -M^{(1)}\Delta_2 h^{(0)} - N\Delta_2^2 h^{(0)} + \frac{1}{6} M_c''\Delta_2 (h^{(0)})^3 \tag{3.1}$$

where the quantities B, $M^{(1)}$ and N are the same as in (2.7).

We restrict ourselve to the case $M_c'' > 0$, which corresponds to the minimum point $H = H_2$ of the dependence of M_c on H; in the other case the stationary solution is unstable, as for $M_c' \neq 0$. We make a change of variables in the first three formulae in (2.8), and put

$$Z = ({}^1/_6 M_c'')^{1/2} h^{(0)} \tag{3.2}$$

Then (3.1) takes the form of the Cahn-Hilliard equation (see e.g. /4/):

$$\partial Z/\partial\tau + \Delta_2^2 Z + M^{(1)}\Delta_2 Z - \Delta_2 Z^3 = 0 \tag{3.3}$$

Bounded solutions satisfying condition (2.11) are of some interest. In the region $M^{(1)} < 0$ the equation has no non-zero solutions. For $M^{(1)} > 0$ one can construct a family of spatially periodic motions in the form of rollers

$$Z = \left(\frac{2q^2 M^{(1)}}{q^2 + 1}\right)^{1/2} \mathrm{sn}\left[\left(\frac{M^{(1)}}{q^2 + 1}\right)^{1/2} (X - X_0)\right], \quad 0 < q < 1 \tag{3.4}$$

(q is the modulus of the Jacobi elliptic function) with period

$$l = \frac{2\pi}{k}, \quad k = \left(\frac{M^{(1)}}{q^2+1}\right)^{1/2} \frac{\pi}{2K(q)} \tag{3.5}$$

It is known /4-6/ that the only stable stationary solutions of (3.3) are the non-periodic solutions

$$Z = \pm (M^{(1)})^{1/2} \operatorname{th}[({}^1/_2 M^{(1)})^{1/2}(X - X_0)] \tag{3.6}$$

A solution of type (3.6) is a stationary "step" sustained by a convective vortex centred on $X = X_0$. At large distances from the step the respective thicknesses of the lower layer are given by the expression

$$H_\pm = H_2 \pm (6\,(M - M_c(H_2))/M_c'')^{1/2}$$

Noting that for values of H close to H_2 the function $M_c(H)$ is approximately described by the formula

$$M_c(H) = M_c(H_2) + {}^1/_2 M_c''(H - H_2)^2$$

one can verify that $M < M_c(H_\pm)$. Hence this convection instability does not develop far from the step.

A solution of type (3.4) for small k is a collective of steps separated from one another. Although such structures are unstable, the growth increment of perturbations for motion with $k/k_* \ll 1$ is exponentially small, so that the process of disruption of such a structure can be very prolonged.

4. Special consideration is required in the case $|H - H_2| \sim \varepsilon^{1/2}$, for which $M_c' = M_c''(H - H_2) \sim \varepsilon^{1/2}$ and the quadratic and cubic terms are of the same order in the evolution equation for the displacement of the surface:

$$\begin{gathered} B\partial h^{(0)}/\partial \bar t = -M^{(1)}\bar\Delta_2 h^{(0)} - N\bar\Delta_2^2 h^{(0)} + {}^1/_2 M_c'' H^{(0)}\bar\Delta_2 h^{(0)2} + \\ {}^1/_6 M_c''\bar\Delta_2 h^{(0)3}, \quad H^{(0)} = (H - H_2)/\varepsilon^{1/2} \end{gathered} \tag{4.1}$$

(where the notation is the same as in (3.1)). Changing variables in the first three formulae in (2.9) and assuming

$$Z = ({}^1/_6 M_c'')^{1/2}(h^{(0)} + H^{(0)})$$

we reduce (4.1) to the form

$$\begin{gathered} \partial Z/\partial\tau + \Delta_2^2 Z + \bar M^{(1)}\Delta_2 Z - \Delta_2(Z^3) = 0 \\ \bar M^{(1)} = M^{(1)} + {}^1/_2 M_c'' H^{(0)2} = (M - M_c(H_2))/\varepsilon \end{gathered} \tag{4.2}$$

The conservation condition for the mean thicknesses of both liquid layers gives, instead of (2.11),

$$\lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} Z\,dX = \bar Z, \quad \bar Z = \left(\frac{M_c''}{6}\right)^{1/2} H^{(0)} \tag{4.3}$$

For stationary one-dimensional solutions we have

$$d^2Z/dX^2 + \bar M^{(1)} Z - Z^3 = CX + D \tag{4.4}$$

where C and D are constants. From the boundedness condition on the solution as $X \to \pm\infty$ we must put $C = 0$. It is clear that finding stationary convective structures is formally equivalent to the problem of non-linear oscillations of a particle with potential energy

$$U(Z) = {}^1/_2 \bar M^{(1)} Z^2 - {}^1/_4 Z^4 - DZ$$

(the role of time is played by the X coordinate). Bounded solutions for $\bar Z \neq 0$ exist in the region of values of the parameter

$$\bar M^{(1)} > 0, \quad 0 < D < 2\,({}^1/_3 \bar M^{(1)})^{3/2}$$

in which the potential $U(Z)$ has the form shown in Fig.4. Of course, for fixed $\bar M^{(1)}$ and D the problem has a family of periodic solutions for which the quantity $E = {}^1/_2(dZ/dX)^2 + U(Z)$ lies in the interval $E_- < E < E_+$, and an aperiodic solution with $E = E_+$. These soltuions can be expressed in terms of elliptic functions /4/. We remark that all solutions with $\bar Z \neq 0$

are non-monotonic, unlike (3.6).

All the above-mentioned solutions are, however, unstable. This conclusion follows from the results of /5/, according to which the signs of the increments of normal perturbations, given by the boundary-value problem

$$\sigma\psi + \Delta_2^2\psi + \bar{M}^{(1)}\Delta_2\psi - 3\Delta_2(Z^2\psi) = 0, \quad \lim_{X\to\pm\infty} |\psi| < \infty$$

are identical with the signs of the eigenvalues of the problem

$$\Delta_2\Phi + (\bar{M}^{(1)} - 3Z^2)\Phi = \sigma\Phi, \quad \lim_{X\to\pm\infty} |\Phi| < \infty \tag{4.5}$$

Because problem (4.5) always possesses sign-varying solutions

$$\Phi = dZ/dX, \quad \sigma = 0$$

the maximum value $\sigma = \sigma_{max}$ is non-degenerate and corresponds to a sign-constant eigenfunction, from which it follows that $\sigma_{max} > 0$.

In practice, however, the layers always have a finite length. It can be shown that the presence of hard thermally insulating side walls at $X = \pm L$ imposes the boundary conditions

$$X = \pm L: \ \partial Z/\partial X = \partial^2 Z/\partial X^2 = 0 \tag{4.6}$$

For $L \gg 1$ and $\bar{Z} < [\bar{M}^{(1)}]^{1/2}$ problem (4.4), (4.6) possesses two monotonic solutions which can with exponential precision be written in the form (3.6) (with $M^{(1)}$ replaced by $\bar{M}^{(1)}$), where $X_0 = \mp L\bar{Z}$. This solution is stable, as is also the case for $\bar{Z} = 0$.

Thus the stationary profile has the form of a step and exists in the domain $M > M_c(H_2) + {}^1/_6 M_c''(H - H_2)^2$.

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